Algorithms on closure systems and their representations

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Committee			
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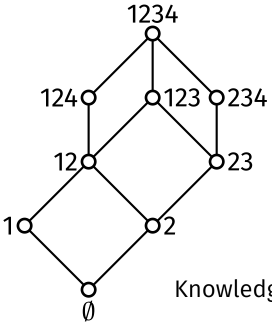
Funded by the **ProFan Project**

- Initial motivation: *Knowledge Space Theory* [Doignon, Falmagne, 1985].
- Some questions of an automated test:
  1. Graphically solve $4x^2 - 3x + 2 = 0$.
  2. Figure out $\frac{\sqrt{4} \times \sqrt{9}}{3} - \frac{6 \times 7}{\sqrt{144}}$.
  3. Compute the discriminant of $3x^2 - x + 8$.
  4. Study the polynomial $7x^2 + 11x - 5$.
- Each question corresponds to a *problem* or *item*:
  1. Graphical resolution.
  2. Arithmetic.
  3. Formula of discriminant.
  4. Study of a 2nd order polynomial.

	1	2	3	4
Wolf	×			
Lil		×	×	
Lazuli		×	×	×
Folavril	×	×		×
Dupont		×		

- Some students took the test!
- Lazuli *masters* item 3.
- $\{2,3\}$ is the *knowledge state* of Lil.

- *Knowledge space* $\mathcal{K}$ over a (finite) collection of items V:

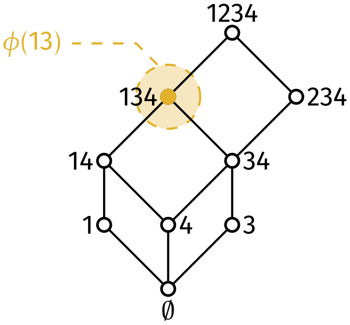


Knowledge space $\mathcal{K}$ (ordered by $\subseteq$)

Definition ▶ Closure system

Closure system $\mathcal{C} \subseteq \mathbf{2}^V$ over V:

- Contains V.
- *Closed by intersection*: $C_1, C_2 \in \mathcal{C}$ entails $C_1 \cap C_2 \in \mathcal{C}$.



$(\mathcal{C}, \subseteq)$

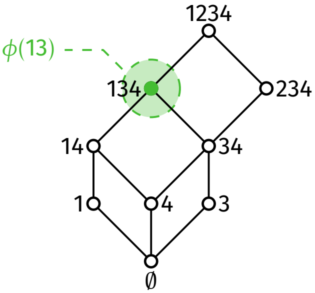
- Sets in $\mathcal{C}$ are *closed sets*.
- $(\mathcal{C}, \subseteq)$ is a *(closure) lattice*.
- Induces a *closure operator* ϕ:
  - $\phi(X)$: minimal closed set including X.
- Closure system = *complement* of Knowledge space!
- $\mathcal{C}$ *standard*: $\phi(v) \setminus \{v\} \in \mathcal{C}$ for each $v \in V$.

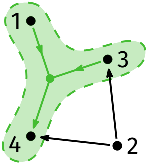
- Closure systems are *ubiquitous* ...
  - *Knowledge Space Theory (KST)*,
  - Formal Concept Analysis (FCA),
  - Propositional logic,
  - Argumentation theory,
  - Databases,
  - ...
- ... but they have *HUGE* size ...
  - If V has n elements, $\mathcal{C}$ can have 2^n closed sets!
- ... and can be *hard to understand*:
  - In KST: asking teachers to provide raw knowledge states is impractical.
- We need *implicit representations!*

Definition ▸ Implications

- *Implication*: expression $A \rightarrow B$, where $A, B \subseteq V$.
- *Implicational base*: set Σ of implications.

- *"If the students fail the items in A, they will fail the items in B".*
- Σ represents a *unique* closure system $\mathcal{C}$.
- $\mathcal{C}$ can be represented by *several (equivalent)* Σ.





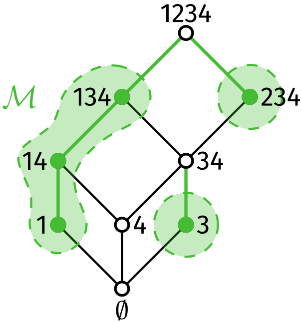
$\Sigma = \{13 \rightarrow 4, 2 \rightarrow 34\}$

Definition ▸ Meet-irreducible elements

Closure system $\mathcal{C}$ over V:

- $M \in \mathcal{C} \setminus \{V\}$ *meet-irreducible* if $M = C_1 \cap C_2$ implies $M = C_1$ or $M = C_2$, $C_1, C_2 \in \mathcal{C}$.
- $\mathcal{M}$ collection of all meet-irreducible elements of $\mathcal{C}$.

- $\mathcal{C}$ *fully recovered* from $\mathcal{M}$ by taking intersections.
- $\mathcal{M}$ is the *"core"* of $\mathcal{C}$.
- $M \in \mathcal{M}$ iff unique *cover*.



Question	Σ	$\mathcal{M}$	$\mathcal{C}$
is v in a min. generator of u?	✗	✓	✓
is P pseudo-closed?	✓	✗	✓
is $\mathcal{C}$ join-semidistributive?	?	✓	✓
Relative size			
size of ... w.r.t. Σ	—	$\exp(\lvert\Sigma\rvert)$	$\exp(\lvert\Sigma\rvert)$
size of ... w.r.t. $\mathcal{M}$	$\exp(\lvert\mathcal{M}\rvert)$	—	$\exp(\lvert\mathcal{M}\rvert)$
size of ... w.r.t. $\mathcal{C}$	$\leq \lvert\mathcal{C}\rvert \times \lvert V\rvert$	$\leq \lvert\mathcal{C}\rvert$	—

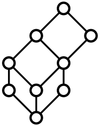
✓ Polynomial

✗ NP-complete

Implications

$2 \rightarrow 4, \quad 13 \rightarrow 4,$
$2 \rightarrow 3, \quad \dots$

Closure system

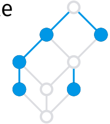


Same knowledge,
Different representations,
Two problems:

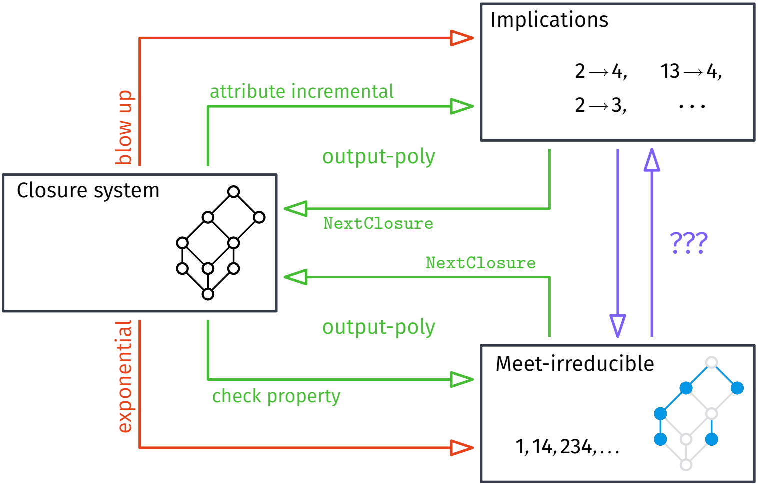
1. TRANSLATION task.
2. FORBIDDEN PAIRS in $\mathcal{C}$.

Meet-irreducible

$1, 14, 234, \dots$



First problem ▸ Translating between the representations of a closure system.


Implications
$2 \rightarrow 4$, $13 \rightarrow 4$,
$2 \rightarrow 3$, ...
blow up
attribute incremental
output-poly
Closure system
NextClosure
NextClosure
???
output-poly
exponential
check property
Meet-irreducible
$1, 14, 234, \ldots$

Problem ▶ Enum. Meet-irr. elements (CCM)

- *Input:* an implicational base Σ for a closure system $\mathcal{C}$ over V.
- *Output:* the meet-irreducible $\mathcal{M}$ of $\mathcal{C}$.

- Surveys by [Bertet et al., 2018], [Wild, 2017].
- Hardness results:
  - *Unknown* complexity.
  - Harder than *hypergraph dualization* (MISEnum), [Khardon, 1995].
  - Enumerating co-atoms is intractable (dualization), [Kavvadias et al., 2000].
- Positive results:
  - General (exponential) algorithms [Mannila, Räihä, 1992], [Wild, 1995].
  - Tractable cases: meet-semidistributive, types of convex geometries [Beaudou et al., 2017], [Defrain, Nourine, V., 2021].

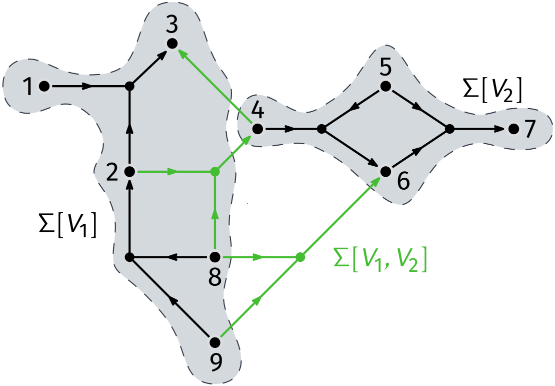
- Strategy:
  - Hierarchical decomposition of Σ.
  - Recursive construction of $\mathcal{M}$.

Definition ▸ Split

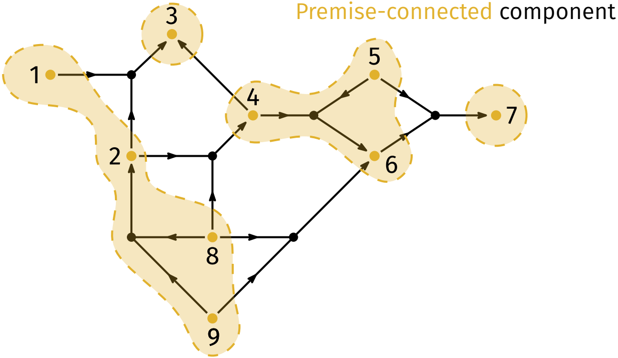
Σ implicational base over V:

- *Split* of Σ: bipartition (V_1, V_2) of V such that $A \subseteq V_1$ or $A \subseteq V_2$ for every $A \longrightarrow B \in \Sigma$.

- Split (V_1, V_2) partitions Σ:
  - $\Sigma[V_1]$ implications included in V_1, with induced $\mathcal{C}_1$, $\mathcal{M}_1$.
  - $\Sigma[V_2]$ implications included in V_2, with induced $\mathcal{C}_2$, $\mathcal{M}_2$.
  - $\Sigma[V_1, V_2]$ implications from V_1 to V_2 or from V_2 to V_1.

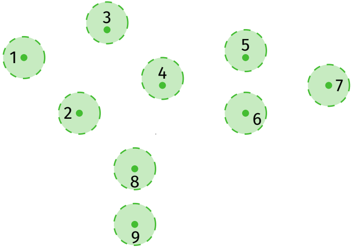


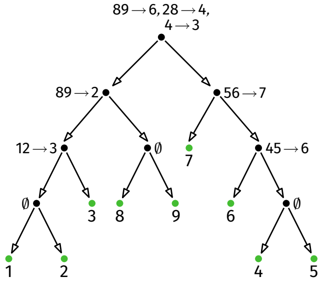
3
1
4
5
$\Sigma[V_2]$
7
2
6
$\Sigma[V_1]$
8
$\Sigma[V_1, V_2]$
9



Proposition ▸ Recognizing splits

Σ has a split (V_1, V_2) if and only if it is not premise-connected.

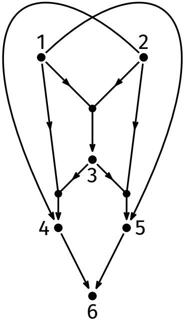




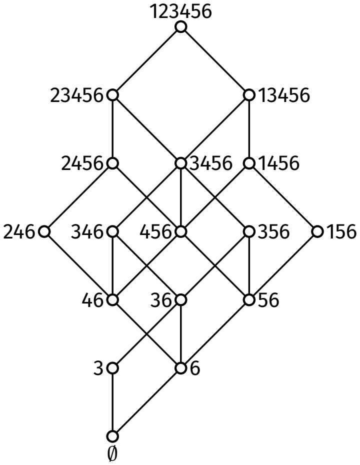
Σ-tree

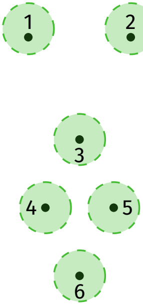
Theorem ▸ Nourine, V.

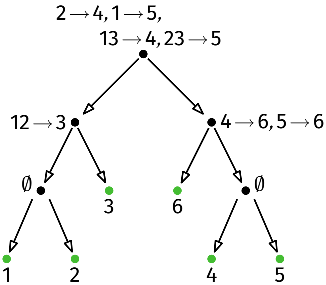
Let Σ be an implicational base over V. A Σ-tree can be computed in *polynomial time and space* in the size of Σ, if it exists.



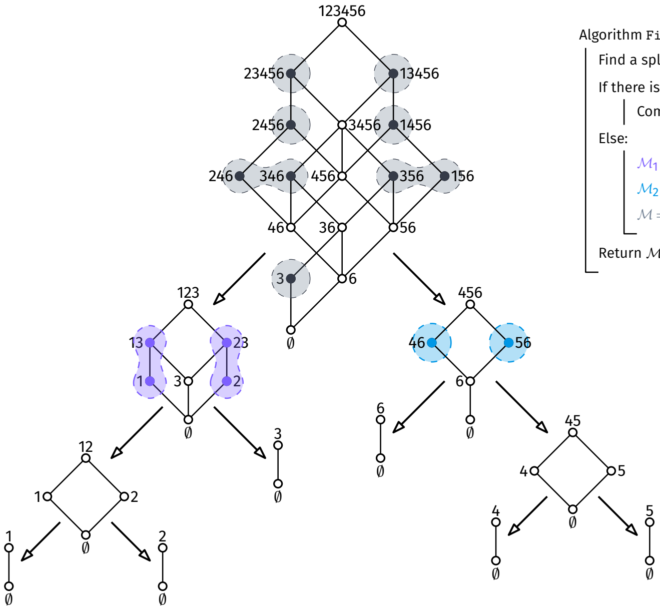
$$\Sigma = \left\{ \begin{array}{lll} 12 \rightarrow 3, & 1 \rightarrow 5, & 5 \rightarrow 6, \\ 13 \rightarrow 4, & 2 \rightarrow 4, & \\ 23 \rightarrow 5, & 4 \rightarrow 6 & \end{array} \right\}$$





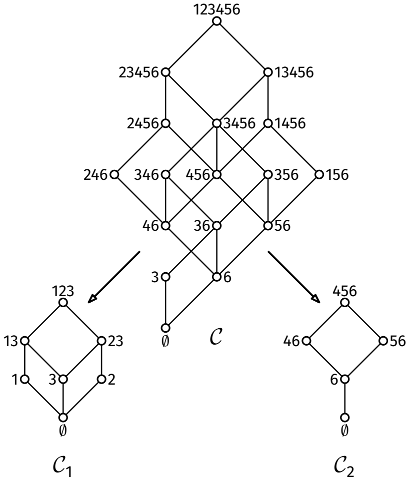


- H-decomposition of Σ implies *H-decomposition* of $\mathcal{C}$.



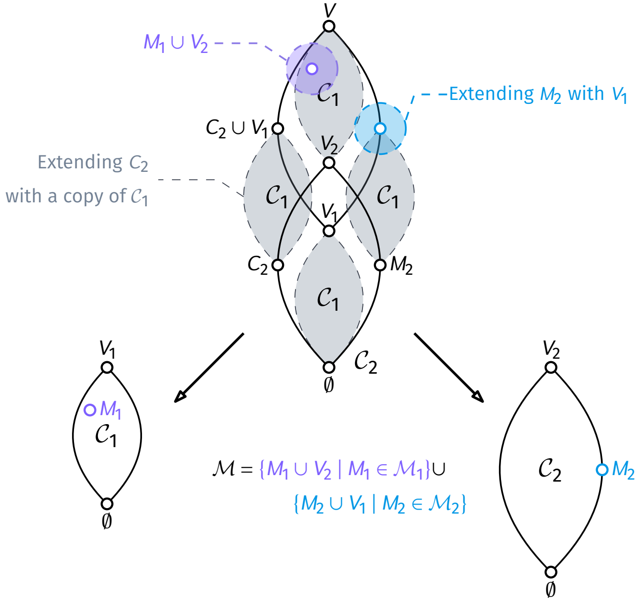
Algorithm $\texttt{FindMeet}(\Sigma, V)$

- Find a split (V_1, V_2) of Σ
- If there is none:
  - Compute $\mathcal{M}$ with another algorithm
- Else:
  - $\mathcal{M}_1 = \texttt{FindMeet}(\Sigma[V_1], V_1)$
  - $\mathcal{M}_2 = \texttt{FindMeet}(\Sigma[V_2], V_2)$
  - $\mathcal{M} = \texttt{ComputeMeet}(\mathcal{M}_1, \mathcal{M}_2, \Sigma)$
- Return $\mathcal{M}$



Observation ▸ Closed sets

- $\mathcal{C} \subseteq \mathcal{C}_1 \times \mathcal{C}_2$.

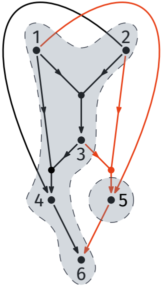


$$\mathcal{M} = \{M_1 \cup V_2 \mid M_1 \in \mathcal{M}_1\} \cup \{M_2 \cup V_1 \mid M_2 \in \mathcal{M}_2\}$$

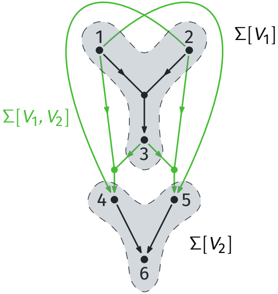
Definition ▸ Acyclic split

Σ an implicational base over V:

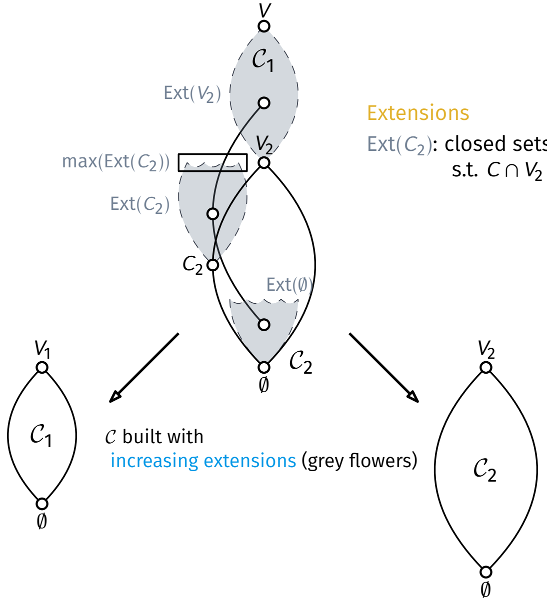
- *Acyclic* split of Σ: split (V_1, V_2) s.t. $A \subseteq V_1$ for each $A \to B \in \Sigma[V_1, V_2]$.



Cyclic split



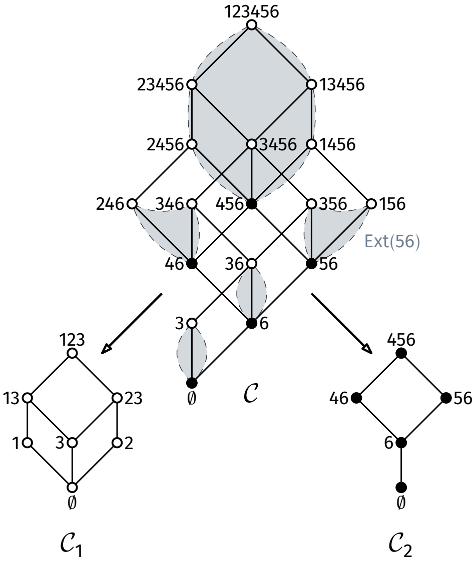
Acyclic split

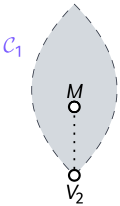


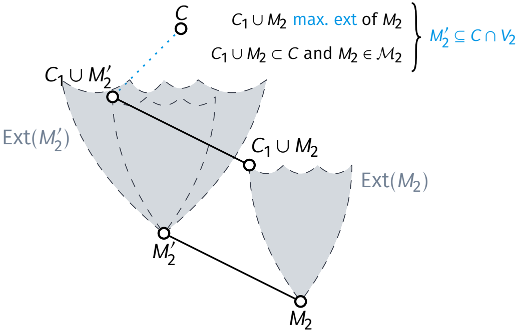
Extensions

$\text{Ext}(C_2)$: closed sets C s.t. $C \cap V_2 = C_2$

$\mathcal{C}$ built with increasing extensions (grey flowers)


123456
23456
13456
2456
3456
1456
246
346
456
356
156
Ext(56)
46
36
56
3
6
∅
$\mathcal{C}$
123
13
23
1
3
2
∅
$\mathcal{C}_1$
456
46
56
6
∅
$\mathcal{C}_2$



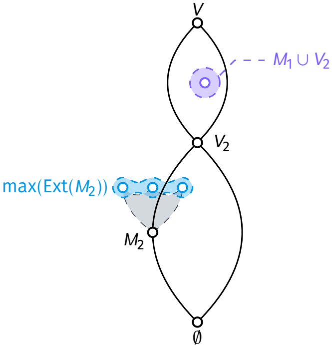


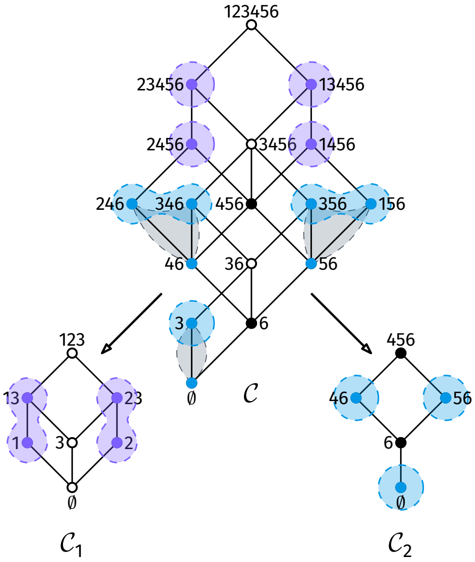
- Case 1: $V_2 \subseteq M$:
  - $C \in \text{Ext}(V_2)$ iff $C = C_1 \cup V_2$ $\quad (C_1 \in \mathcal{C}_1)$.
  - $M \in \mathcal{M}$ iff $M = M_1 \cup V_2$ $\quad (M_1 \in \mathcal{M}_1)$.
- Case 2: $V_2 \not\subseteq M$:
  - $M \in \text{Ext}(M_2)$, $M_2 \in \mathcal{M}_2$ (increasing extensions).
  - $M \in \max(\text{Ext}(M_2))$ for some $M_2 \in \mathcal{M}_2$.
  - $M \in \mathcal{M}$ iff $M \in \max(\text{Ext}(M_2))$ $\quad (M_2 \in \mathcal{M}_2)$.

Theorem ▶ Nourine, V.

Let Σ be an implicational base over V with acyclic split (V_1, V_2). Then $|\mathcal{M}| \geq |\mathcal{M}_1| + |\mathcal{M}_2|$ and:

$$\mathcal{M} = \{M_1 \cup V_2 \mid M_1 \in \mathcal{M}_1\} \cup \{C \in \max(\text{Ext}(M_2)) \mid M_2 \in \mathcal{M}_2\}$$

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$\mathcal{C}$
123
13
23
1
3
2
∅
$\mathcal{C}_1$
456
46
56
6
∅
$\mathcal{C}_2$

Algorithm $\texttt{FindMeet}(\Sigma, V)$

Find an acyclic split (V_1, V_2) of Σ

If there is none:

  Compute $\mathcal{M}$ with another algorithm

Else:

  $\mathcal{M}_1 = \texttt{FindMeet}(\Sigma[V_1], V_1)$

  $\mathcal{M}_2 = \texttt{FindMeet}(\Sigma[V_2], V_2)$

  $\mathcal{M} = \{M_1 \cup V_2 \mid M_1 \in \mathcal{M}_1\}$

  For each $M_2 \in \mathcal{M}_2$:

    $\mathcal{M} = \mathcal{M} \cup \max(\mathrm{Ext}(M_2))$

Return $\mathcal{M}$

- *Beware*:
  1. Size of $\mathcal{M}_1, \mathcal{M}_2$? ✓
  2. ~~Complexity of ComputeMeet~~?
  3. Complexity of *finding extensions*.

Problem ▸ Computing Maximal Extension (MaxExt)

- *Input:* implicational base Σ with acyclic split (V_1, V_2), $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) the meet-irreducible elements associated to $\Sigma[V_1]$ (resp. $\Sigma[V_2]$), a closed set C_2 of $\Sigma[V_2]$.
- *Output:* $\max(\text{Ext}(C_2))$.

- $\max(\text{Ext}(C_2))$ has a *dual* antichain in $\mathcal{C}_1$ coded by $\Sigma[V_1, V_2]$.
- MaxExt is then *equivalent* to dualization with $\mathcal{M}$ *and* Σ.
- If $\Sigma[V_1] = \emptyset$, MaxExt is equivalent to MISEnum.

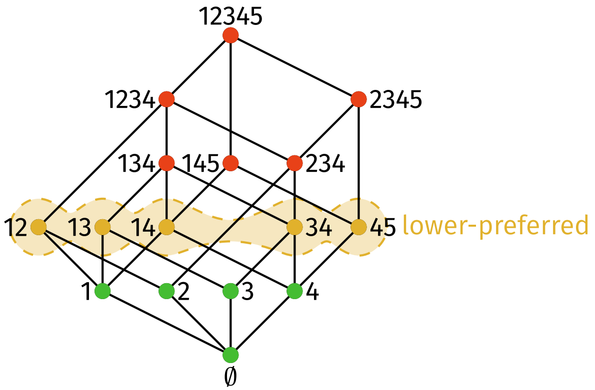
Corollary ▸ Applications

Let Σ be an implicational base over V. Assume there exists a full partition $V_1, \ldots, V_k$ of V such that for every implication $A \rightarrow b \in \Sigma$, $A \subseteq V_i$ and $b \in V_j$ for some $1 \leq i < j \leq k$. Then CCM can be solved in *output-quasipolynomial time*.

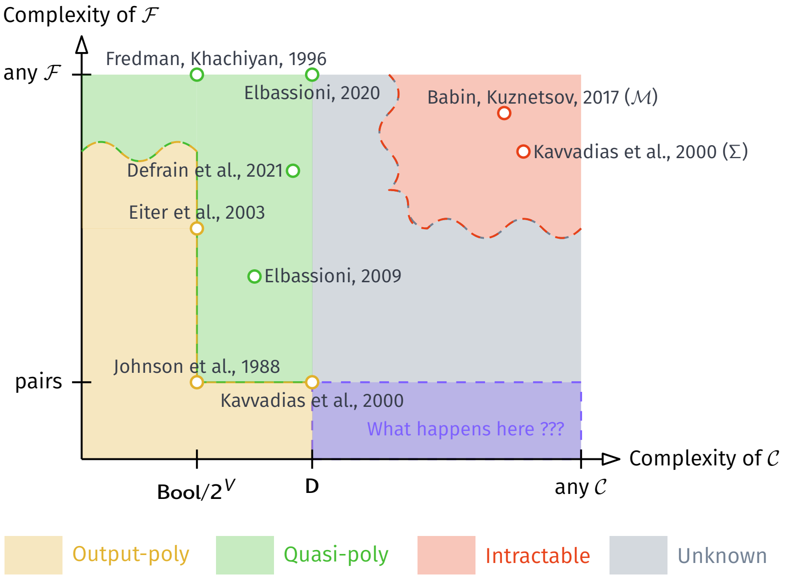
- Particular case of *acyclic convex geometry* [Adaricheva, 2017], [Hammer, Kogan, 1995].
- Generalizes *ranked convex geometry* [Defrain, Nourine, V., 2021], where CCM is *equivalent to* MISEnum.
- Also works for "*simple closure systems*" (diamonds, pentagons, etc).

- Problem:
  - CCM: enumerating meet-irreducible elements from implications.
  - *Unknown complexity*, *harder than* MISENUM.
- Results:
  - (Acyclic) split operation.
  - Hierarchical decomposition of Σ, recursive construction of $\mathcal{M}$.
  - New tractable cases (*output-quasipolynomial time*) in acyclic convex geometries.
- Further research:
  - *Recognition* of an acyclic split from $\mathcal{M}$?
  - Generalization to *"simple" non-acyclic splits*?
  - Complexity of CCM in (acyclic) convex geometries?

Second problem ▸ Forbidden pairs in closure systems



- Closure system $\mathcal{C}$ (given by Σ or $\mathcal{M}$), *forbidden sets* $\mathcal{F} = \{134, 15, 24\}$.
- 1 is *lower-admissible* : does not contain a set in $\mathcal{F}$.
- 12 is *lower-preferred* : *inclusion-wise max.* lower-admissible.


Complexity of $\mathcal{F}$
any $\mathcal{F}$
pairs
Fredman, Khachiyan, 1996
Elbassioni, 2020
Babin, Kuznetsov, 2017 ($\mathcal{M}$)
Kavvadias et al., 2000 (Σ)
Defrain et al., 2021
Eiter et al., 2003
Elbassioni, 2009
Johnson et al., 1988
Kavvadias et al., 2000
What happens here ???
Complexity of $\mathcal{C}$
Bool/2^V
D
any $\mathcal{C}$
Output-poly
Quasi-poly
Intractable
Unknown

Definition ▶ lower-preferred closed sets

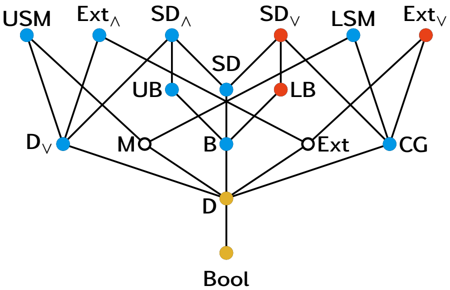
$\mathcal{C}$ closure system over V, family $\mathcal{F}$ over V of *forbidden pairs* for $\mathcal{C}$:

- *Lower-admissible* if $F \not\subseteq C$ for each $F \in \mathcal{F}$.
- *Lower-preferred* if inclusion-wise max. lower-admissible.

Problem ▶ Enum. Lower-Pref. with forb. Pairs (ELP-P(α))

- *Input:* a representation α for a closure system $\mathcal{C}$, a family $\mathcal{F}$ of *forbidden pairs* (both over V).
- *Output:* *lower-preferred* closed sets of $\mathcal{C}$ w.r.t. $\mathcal{F}$.

- Models inconsistency:
  - *Poset + forbidden pairs*: representation for median semilattices [Barthélemy, Constantin, 1993].
  - *implications + forbidden pairs*: representation for modular semilattices [Hirai, Nakashima, 2018].



Bool = Boolean

D = Distributive

SD = Semidistributive

Ext = Extremal

M = Modular

SM = Semimodular

B = Bounded

CG = Convex Geometry

● Output-poly ● $\geq$ DUAL(α) in **CG**, **B** or **D**$_\vee$ ● ELP-P(Σ) hard

Corollary ▸ Nourine, V.

The problem ELP-P(α) is *intractable*. Moreover, ELP-P(Σ) is *intractable* in *lower-bounded* and *join-extremal* closure systems.

Definition ▶ Minimal Generator, Carathéodory number

$\mathcal{C}$ (standard) closure system over V:

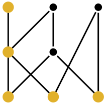
- $A \subseteq V$ *minimal generator* of $u \in V$: $u \in \phi(A)$ and $u \notin \phi(A')$, $\forall A' \subset A$.
- *Carathéodory number* $\mathrm{cc}(\mathcal{C})$ of $\mathcal{C}$: *maximal size* of a minimal generator.

Theorem ▶ Nourine, V.

The problem ELP-P(α) can be solved in:

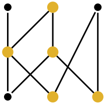
- *Output-polynomial time* if $\mathrm{cc}(\mathcal{C}) \leq k$, for some constant $k \in \mathbb{N}$.
- *Output-quasipolynomial time* if $\mathrm{cc}(\mathcal{C}) \leq \log(|V|)$.

- Closure systems where $\mathrm{cc}(\mathcal{C})$ is constant:



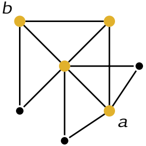
ideals of a poset

$\mathrm{cc}(\mathcal{C}) = 1$



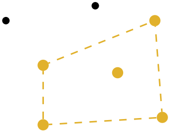
convex subsets of a poset

$\mathrm{cc}(\mathcal{C}) \leq 2$



geod. conv. of a chordal graph

$\mathrm{cc}(\mathcal{C}) \leq 2$



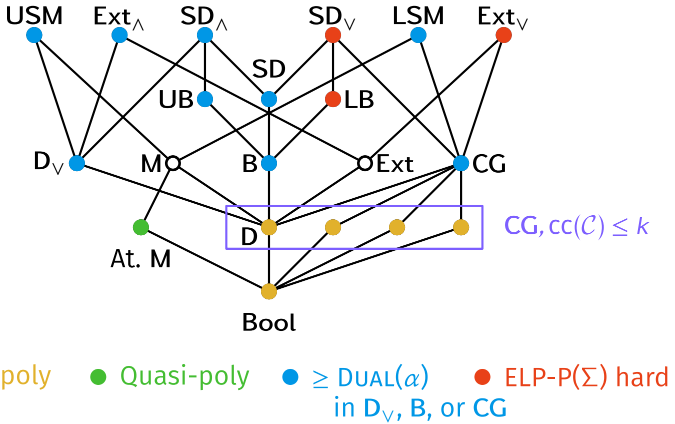
convex hull in $\mathbb{R}^k$

$\mathrm{cc}(\mathcal{C}) = k + 1$

- Biatomicity [Bennett, 1987] + Independence criterion [Grätzer, 2011].

Corollary ▸ Nourine, V.

The problem ELP-P(α) can be solved in *output-quasipolynomial time* in *atomistic modular closure systems*.

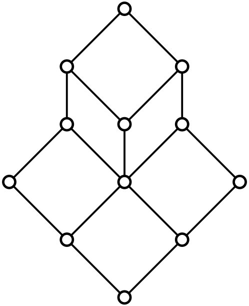


- Further research:
  - Complexity of ELP-P(α) in *modular* and *extremal* closure systems?
  - Characterize the lattices where ELP-P(α) $\equiv$ enumerate max. independent sets of a graph?

- Context:
  - Initial motivation from knowledge spaces (ProFan project).
  - Theoretical study of closure systems and their representations.
- First problem – translating between the representations:
  - *Unknown complexity*, *harder than* MISEnum.
  - *New tractable classes* based on *hierarchical decompositions* of implications.
  - *(Not in this talk)* previous work on *ranked convex geometries*.
- Second problem – closure systems with forbidden sets:
  - Enumerating admissible and preferred closed sets.
  - *Hardness results for* ELP-P(α) using Dual(α), *tractable cases* based on the Carathéodory number.
  - *(Not in this talk)* results for *forbidden supersets*.
- Open questions:
  - What is the complexity of CCM in acyclic convex geometries?
  - Characterize the lattices where ELP-P(α) $\equiv$ max. independent sets of a graph?

- Translation:
  - *The enumeration of meet-irreducible elements based on hierarchical decompositions of implicational bases*. With *Lhouari Nourine*.
    Submitted to *Theoretical Computer Science* and communicated at *WEPA 2020*, *FCA4AI 2020*, *ICTCS 2020*.
  - *Translating between the representations of a ranked convex geometry*. With *Oscar Defrain* and *Lhouari Nourine*.
    Published in *Discrete Mathematics* (2021) and communicated at *WEPA 2019*.
- Forbidden sets:
  - *Enumerating maximal consistent closed sets in closure systems*. With *Lhouari Nourine*.
    Published in *Proceedings of ICFCA 2021* and communicated at *ICFCA 2021*.
- Other:
  - *Towards declarative comparabilities: application to functional dependencies*. With *Lhouari Nourine* and *Jean-Marc Petit*.
    Under review in *Journal of Computer and System Sciences* and communicated at *BDA 2021*.

Thank you for your attention!



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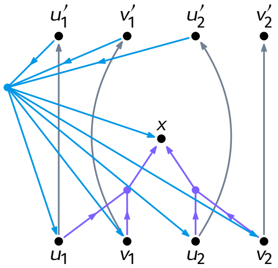
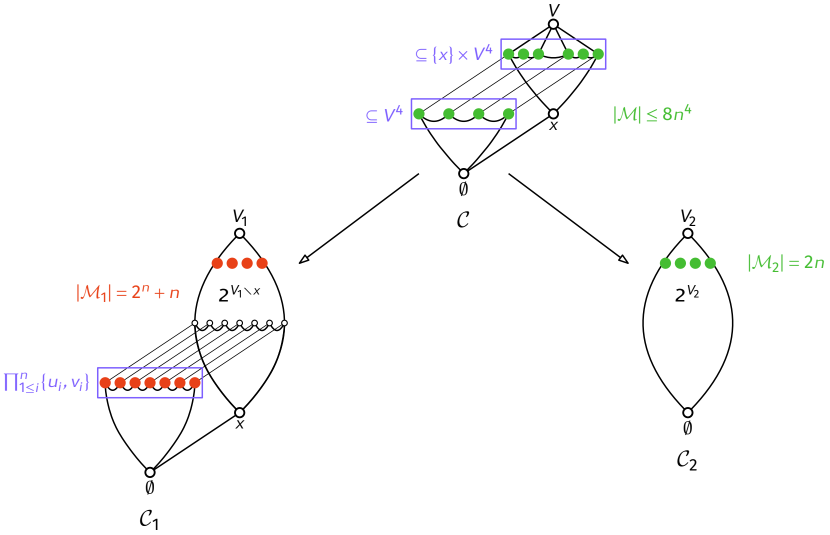
- universe $V = V_1 \cup V_2$ with:
  - $V_1 = \{u_1, \dots, u_n, v_1, \dots, v_n, x\}$, $V_2 = \{u'_1, \dots, u'_n, v'_1, \dots, v'_n\}$, $n \in \mathbb{N}$
- Σ over V with split (V_1, V_2):
  - $\Sigma[V_1] = \{u_i v_i \to x \mid 1 \leq i \leq n\}$, $\Sigma[V_2] = \emptyset$
  - $\Sigma[V_1, V_2] = \{u_i \to u'_i \mid 1 \leq i \leq n\} \cup \{v_i \to v'_i \mid 1 \leq i \leq n\} \cup \{A \to V_1 \mid A \subseteq V_2, |A| = 3\}$





V
$\subseteq \{x\} \times V^4$
$\subseteq V^4$
x
$|\mathcal{M}| \leq 8n^4$
$\emptyset$
$\mathcal{C}$
V_1
$|\mathcal{M}_1| = 2^n + n$
$2^{V_1 \setminus x}$
$\prod_{1 \leq i}^{n} \{u_i, v_i\}$
x
$\emptyset$
$\mathcal{C}_1$
V_2
$|\mathcal{M}_2| = 2n$
2^{V_2}
$\emptyset$
$\mathcal{C}_2$