THE E-BASE OF FINITE (SEMIDISTRIBUTIVE) LATTICES*

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Simon Vilmin

Aix Marseille Université, CNRS, LIS, Marseille, France

Kira Adaricheva

Department of Mathematics, Hofstra University, New-York, USA

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Context: describe a closure system with implications $A \to x$ where A is a "minimal" generator of x

Different meanings of minimality lead to different implications

subset min

order min

order + closure min

Canonical direct base $\supseteq$ D-base $\supseteq$ E-base

We are interested in the E-base

Question: sometimes the E-base is valid, sometimes not ...
so what are the classes of (closure) lattices where it is valid ?

THM (Adaricheva, V., 25+): the E-base of a closure system with semidistributive lattice is valid and minimum

Other (upcoming) results :

- valid : atomistic modular lattices, lattices of binary matroids
- non-valid : geometric, modular and join-distributive lattices
- hardness of computing E-relation from context or implications

PART 1: what is the $\mathcal{E}$-base ?

- some notations
- meanings of minimality
- the $\mathcal{E}$-base

PART 2: is the $\mathcal{E}$-base valid ?

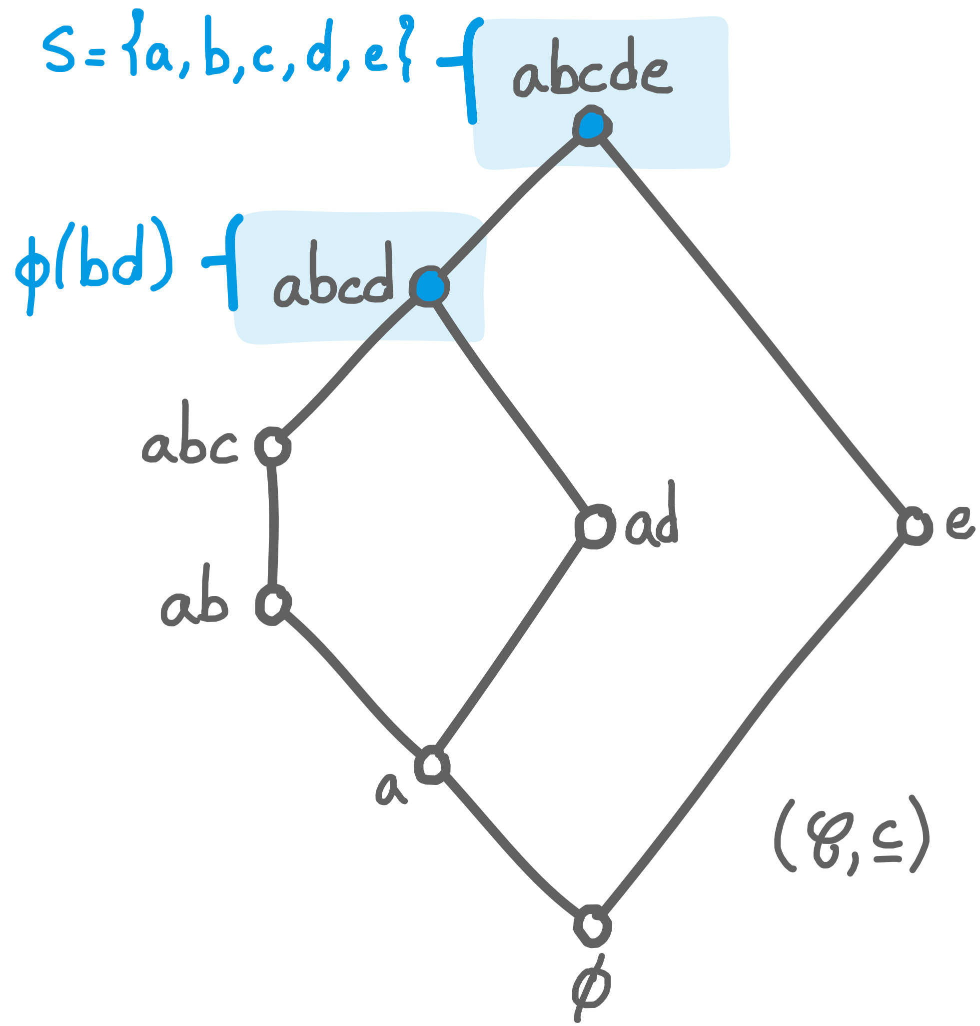
- related work and results
- $\mathcal{E}$-base against canonical base
- $\mathcal{E}$-generators and prime elements

PART 1: what is the E-base?

- closure system $(S, \mathscr{C})$: ground set S, $\mathscr{C} \subseteq 2^S$ contains S and is closed under intersection
- closure operator ϕ
- closure lattice $(\mathscr{C}, \subseteq)$

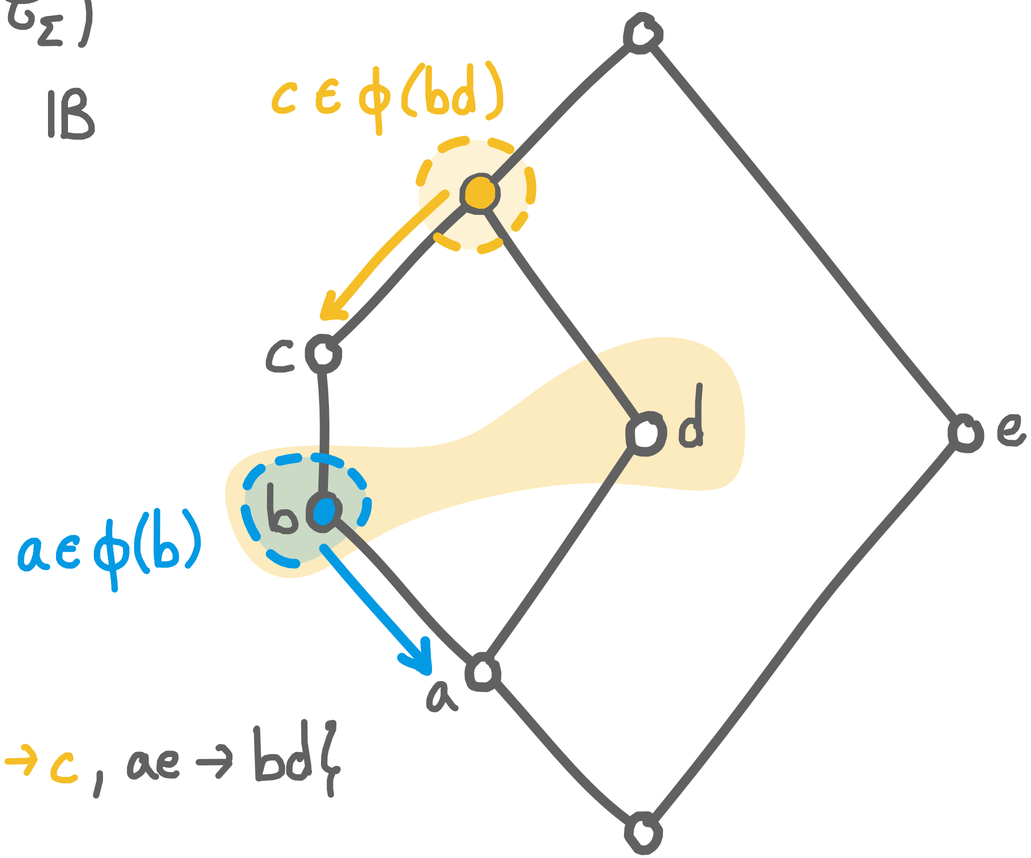
$C = \phi(A)$: A spans C

$X \subseteq \phi(A)$: A generates X



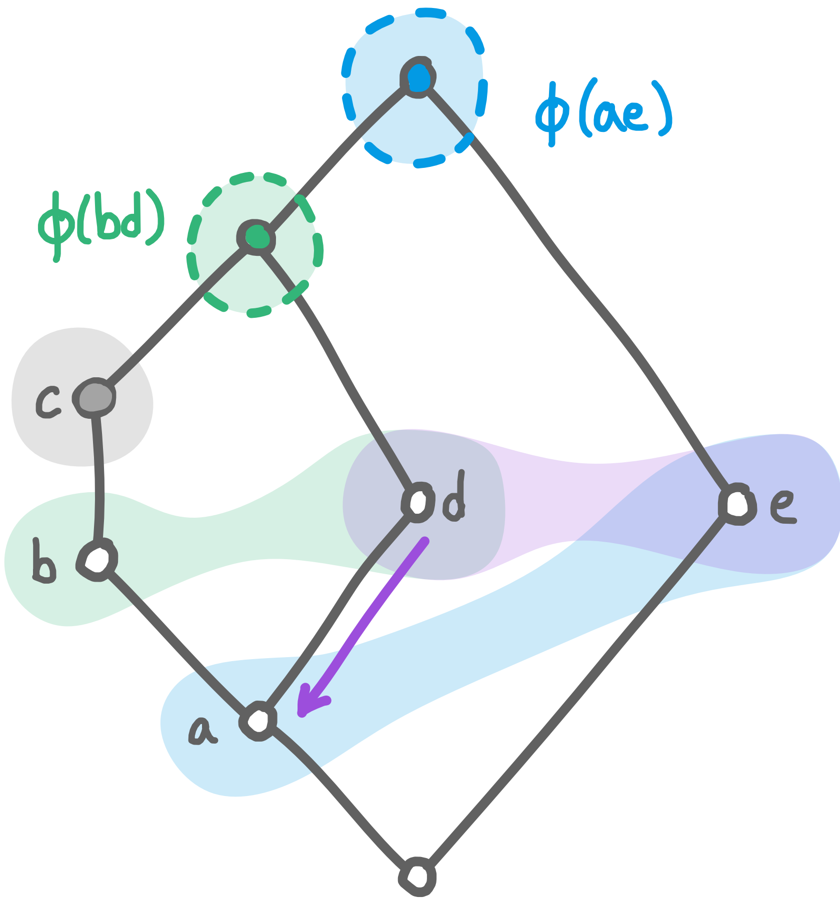
- implicational base (IB) (S, Σ): Σ set of implications $A \to B$ with $A, B \subseteq S$
- associated closure system $(S, \mathscr{C}_\Sigma)$
- each closure system admits ≥ 1 IB

(S, Σ) is a valid IB of $(S, \mathscr{C})$ if $\mathscr{C}_\Sigma = \mathscr{C}$



$\Sigma = \{c \to b,\ b \to a,\ d \to a,\ bd \to c,\ ae \to bd\}$

Flavors of minimality



some "minimal" generators of c

$\boxed{bd, ae, de}$ subset min *

ae is below de

$\boxed{bd, ae}$ order min *

$\phi(bd) \subset \phi(ae)$

$\boxed{bd}$ order + closure min *

* minimal generators * Δ-generators * E-generators

The E-base

DEF: $A \subseteq S$ is a **E-generator** of x if

(1) $x \in \phi(A)$ but $x \notin \phi(a)$, $a \in A$

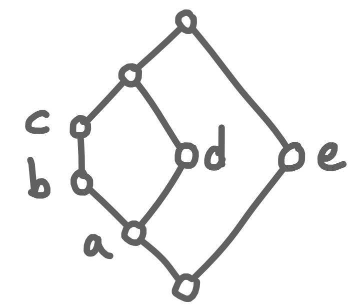
(2) for all $B \subseteq \bigcup_{A} \phi(a)$, $x \in \phi(B) \Rightarrow A \subseteq B$ — order min.

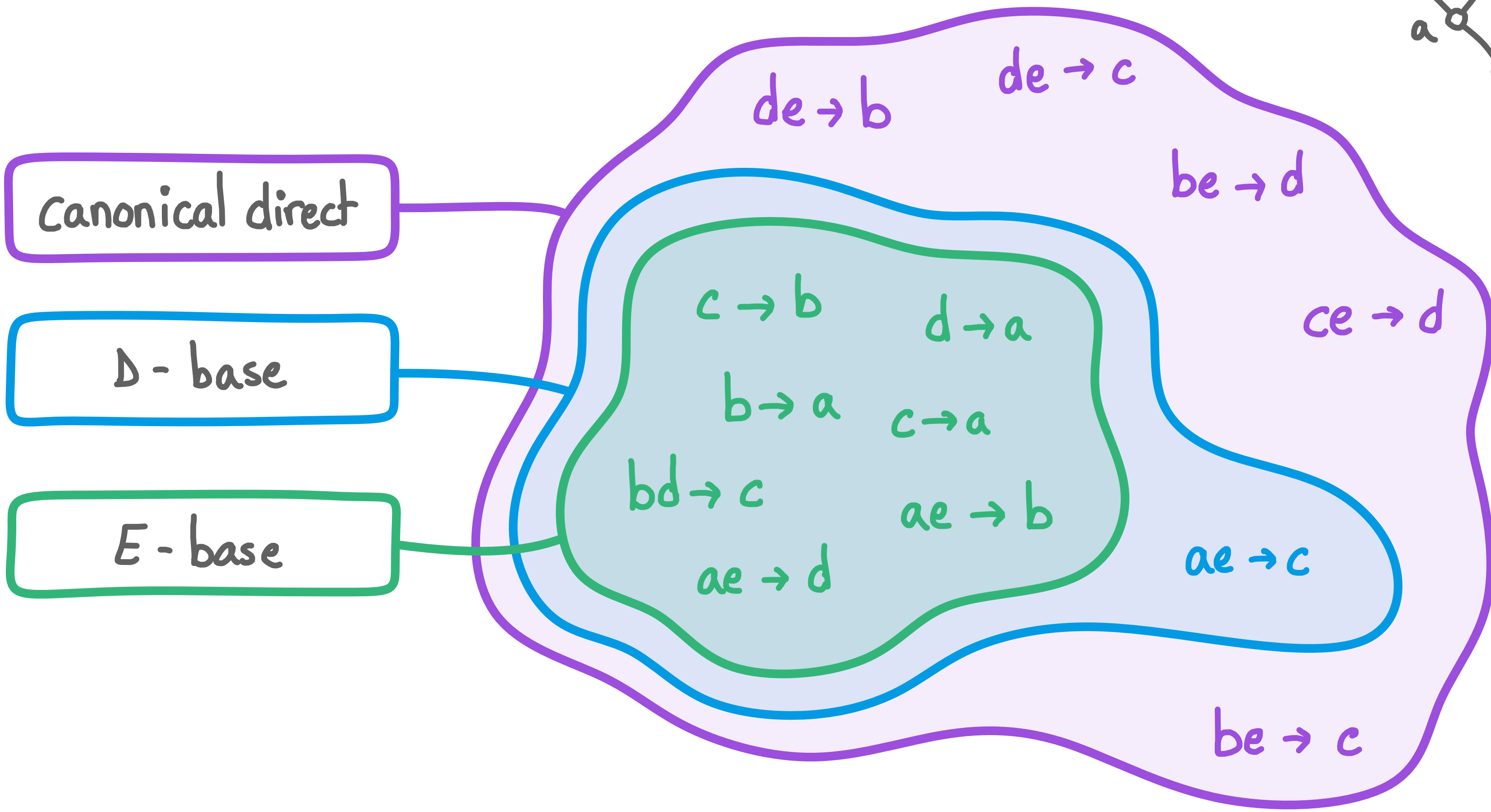
(3) $\phi(A) \in \min_{\subseteq} \{ \phi(A') : A'$ satisfies (1) and (2)$\}$ — closure min.

DEF: the **E-base** of $(S, \mathcal{E})$ is (S, Σ_E) with

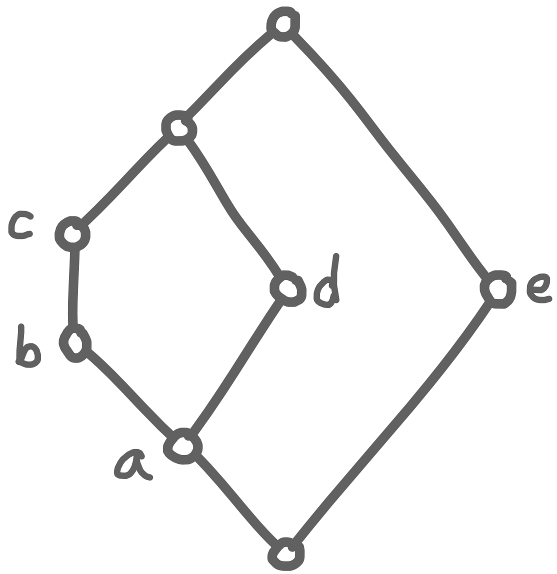
$$\Sigma_E = \{ a \to b : b \in \phi(a) \}$$
$$\cup \{ A \to b : A \text{ is a E-generator of } b \}$$

Back to the example





Is the E-base valid ?



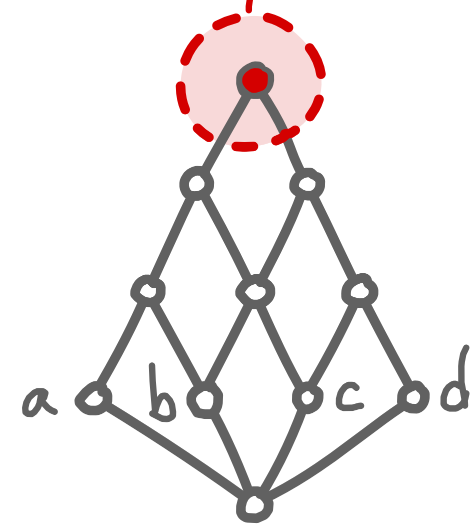
$$\Sigma_E = \{c \to b,\ c \to a,\ b \to a,\ d \to a\}$$
$$\cup \{ae \to b,\ ae \to d,\ bd \to c\}$$

Valid implicational base

$$\Sigma_E = \{ac \to b,\ bd \to c\} \quad ad \to bc\ ?$$

Non-valid implicational base

not correctly described



PART 2: is the E-base valid ?

Origins of the E-base

- E-generators come from free lattices, Freese et al. 1995
- then turned into an IB, Adaricheva et al. 2013

Question: what are the classes of lattices where it is valid?

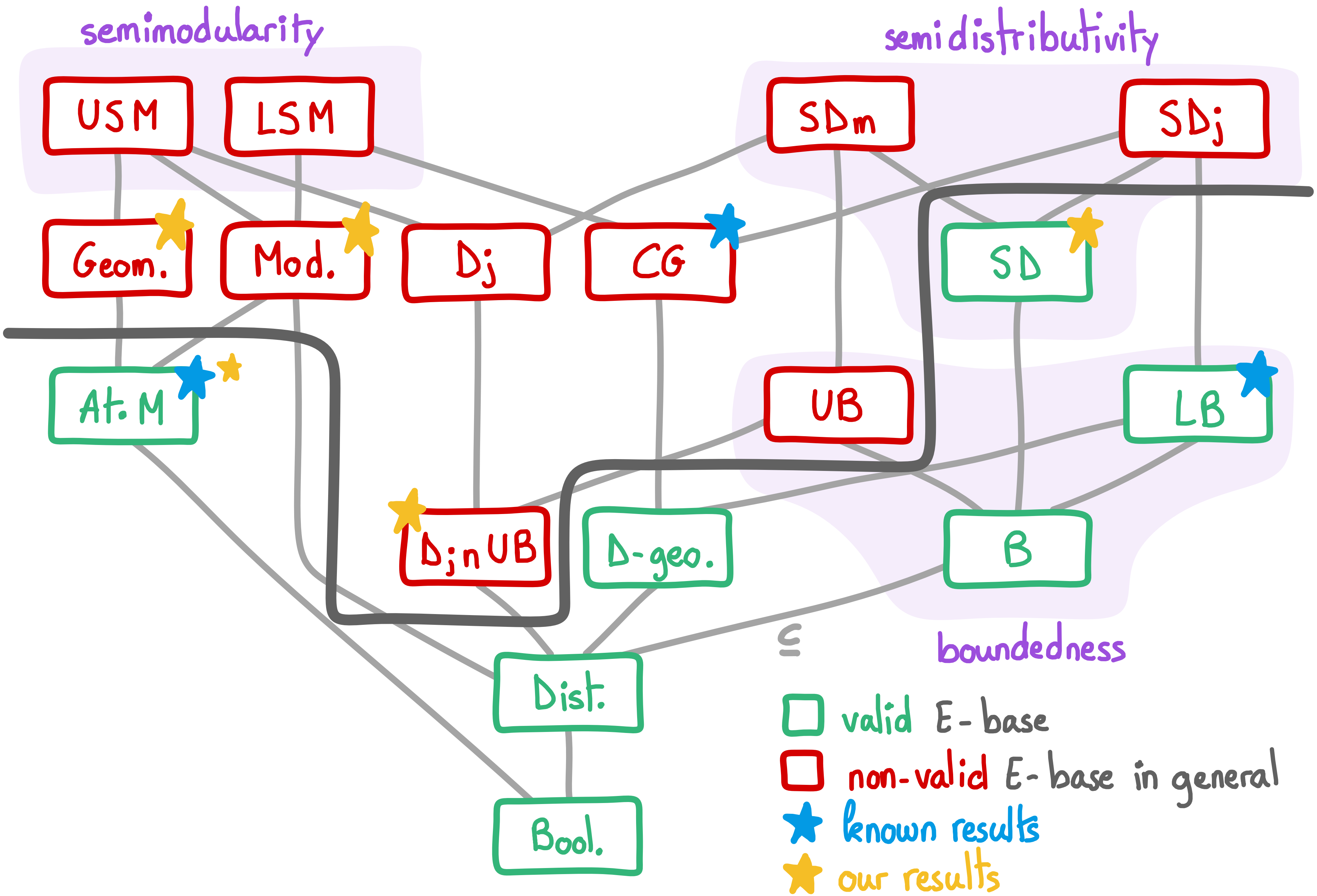
Known results (Adaricheva et al. 2013)

- valid and minimum in lower bounded lattices,
- non-valid in convex geometries in general

Deduced from earlier works (mostly Wild, 1994, Wild, 2000)

- valid in atomistic modular lattices and binary matroids

Classes of lattices with valid E-base



Question : how to study the validity of the E-base ?

(in particular for semidistributive lattices)

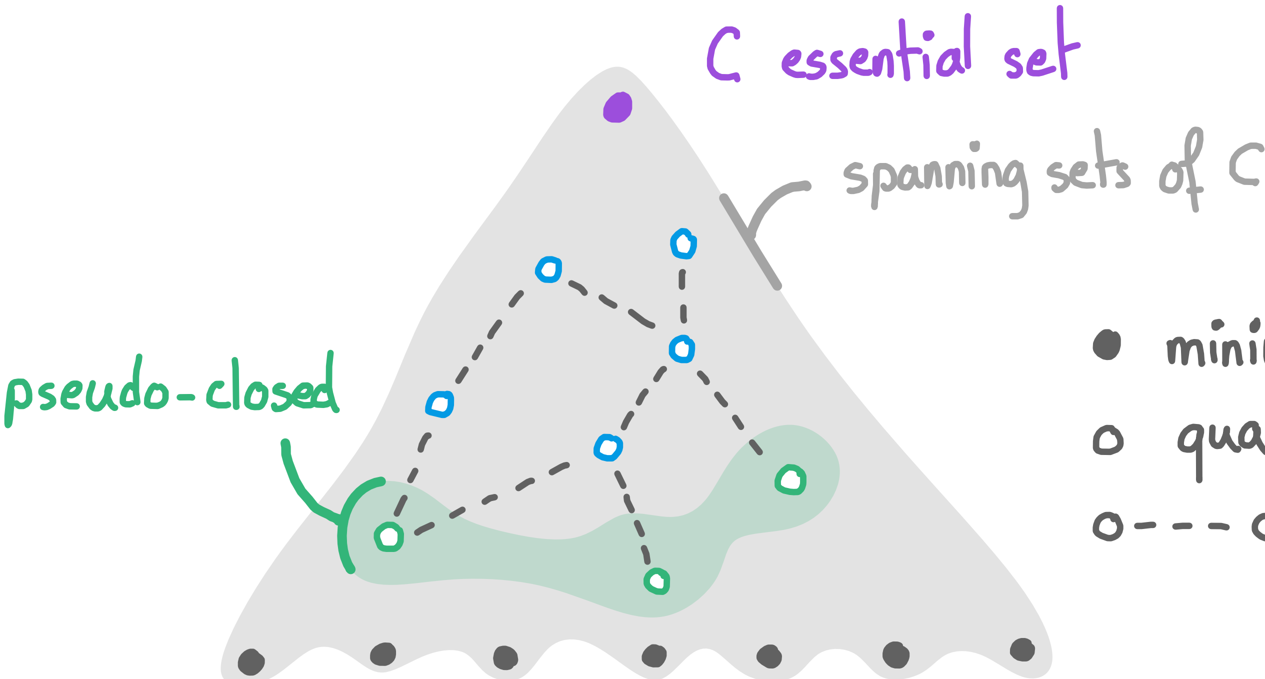
Two ideas :

(1) compare the E-base with the canonical base

(2) find the meaning of E-generators in the lattice in terms of prime elements

Quasi-closed, pseudo-closed, essential sets

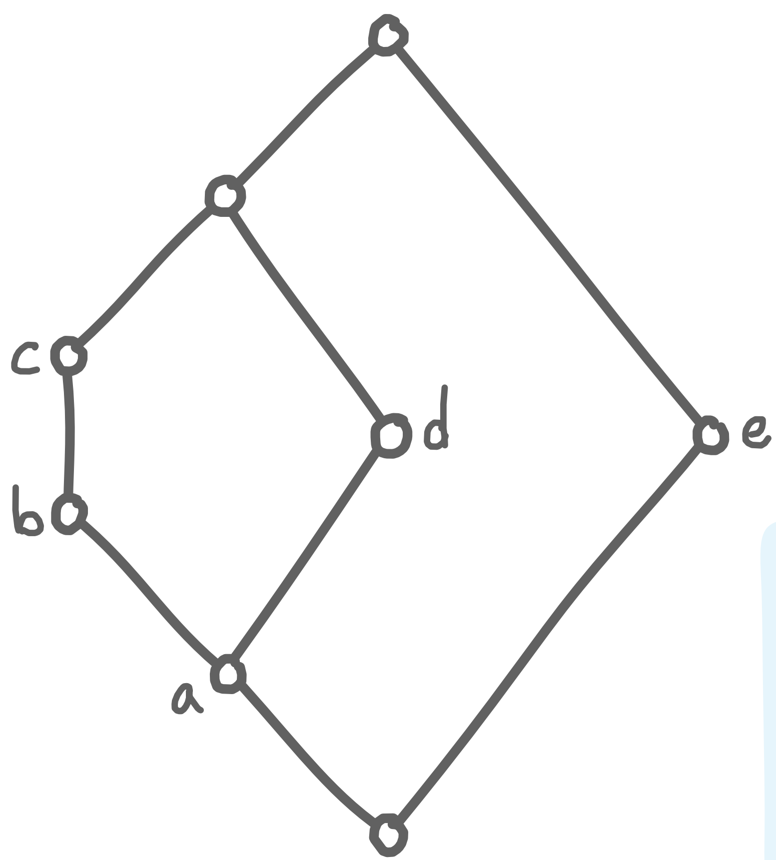
- $Q \subseteq S$ quasi-closed: for $Y \subseteq Q$, $\phi(Y) \subset \phi(Q)$ implies $\phi(Y) \subseteq Q$
- $P \subseteq S$ pseudo-closed: $\subseteq$-min quasi-closed spanning sets of $\phi(P)$
- $C \subseteq S$ essential set: $C = \phi(P)$ for some pseudo-closed set P



- ● minimal spanning set of
- ○ quasi-closed (non-closed)
- ○---○ set-inclusion

Canonical base (see, e.g., Duquenne, Guigues, 1986)

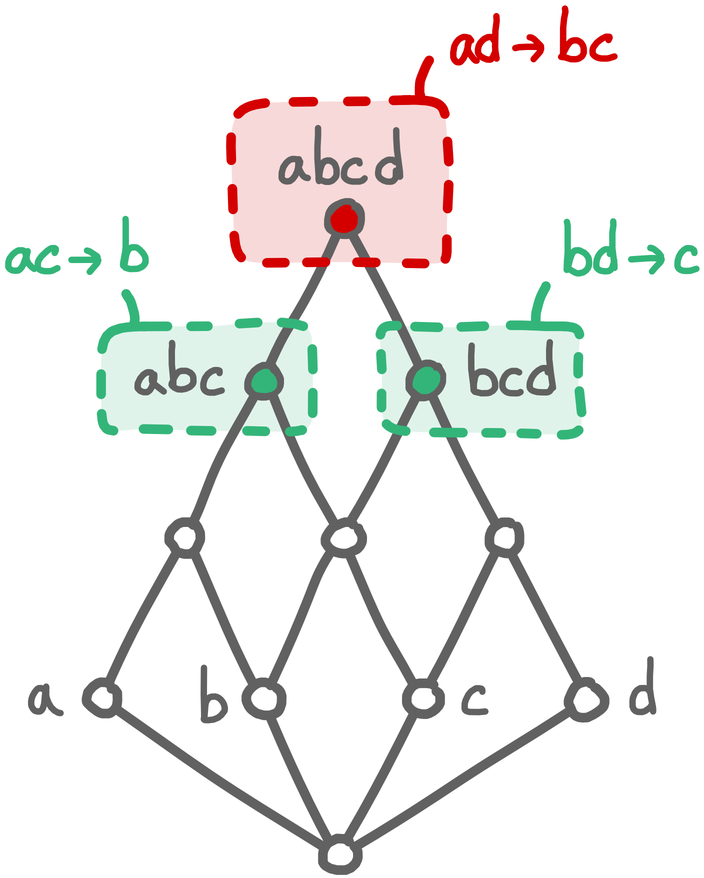
DEF: the canonical base of $(S, \mathscr{C})$ is (S, Σ_c) where
$$\Sigma_c = \{ P \to \phi(P) \setminus P : P \text{ pseudo-closed} \}$$



$$\Sigma_c = c \to ab,$$
$$d \to a,$$
$$ae \to bcd,$$
$$bda \to c$$

THM: any valid IB of $(S, \mathscr{C})$ contains an implication $A \to X$ with $A \subseteq P$ and $\phi(A) = \phi(P)$ for each pseudo-closed set P

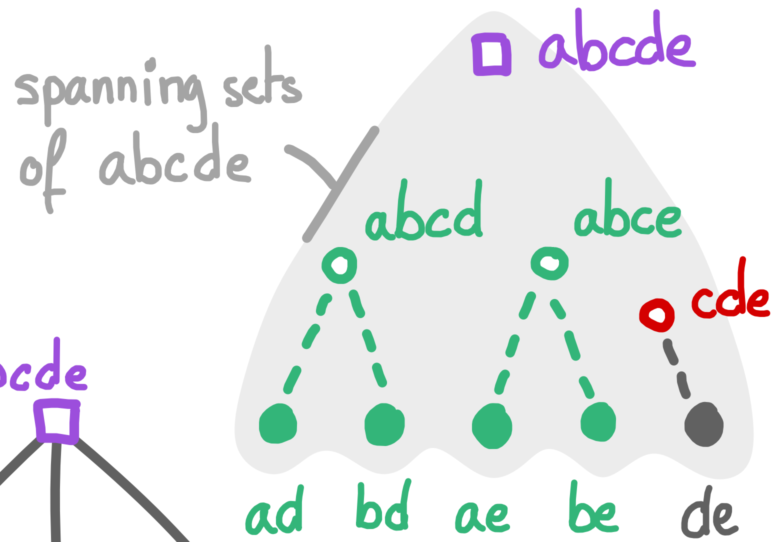
E-base vs. canonical: missing essential sets

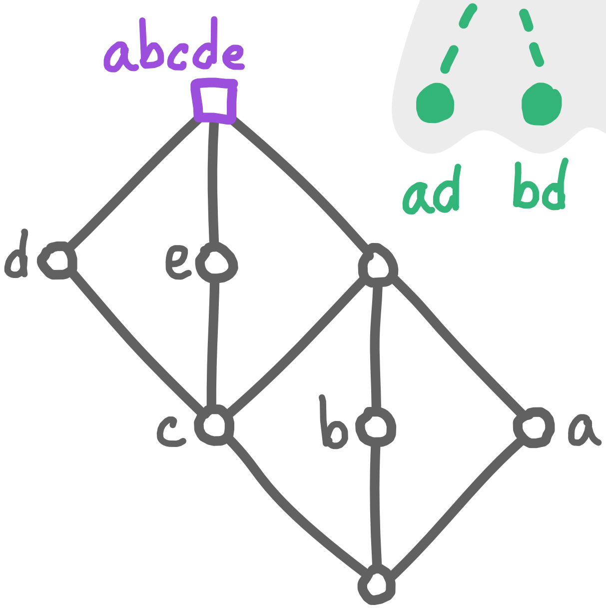


Σ_C	Σ_E
$ac \to b$	$ac \to b$
$bd \to c$	$bd \to c$
$ad \to bc$	

PROB: essential set abcd not the closure of any E-generator

E-base vs. canonical: missing pseudo-closed sets





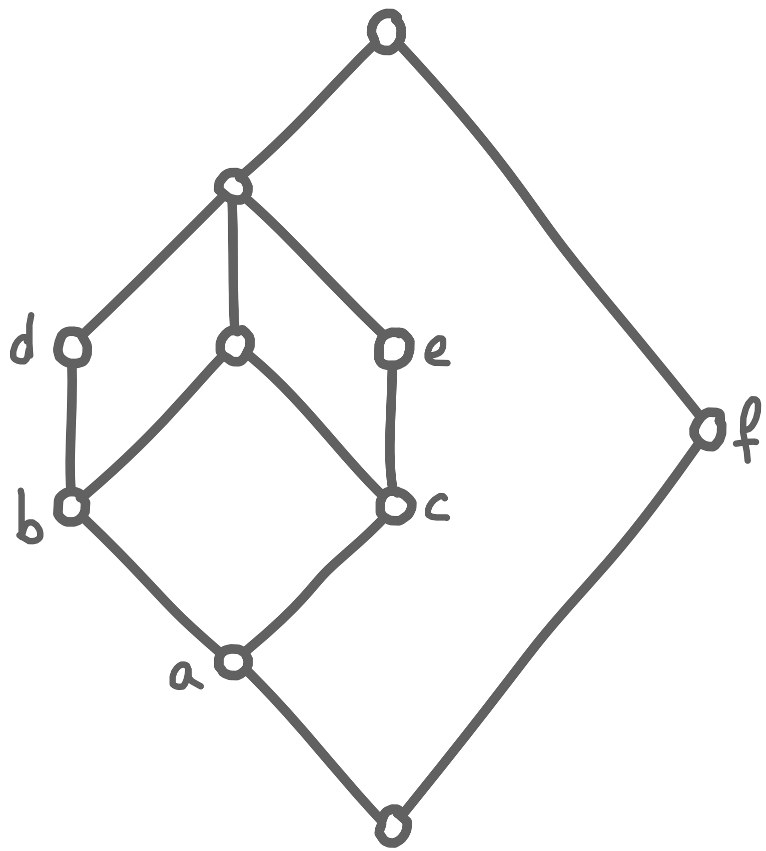
- Each essential set is spanned by some E-generator

- essential set abcde

Σ_C	Σ_E
$abce \to d$	$ae \to d$, $be \to d$
$abcd \to e$	$ad \to e$, $bd \to e$
$cde \to ab$	

PROB: pseudo-closed set cde not subsumed by any E-generator spanning abcde

E-base vs. canonical: not reaching enough elements



- Each pseudo-closed set is subsumed by a E-generator spanning the same essential set.

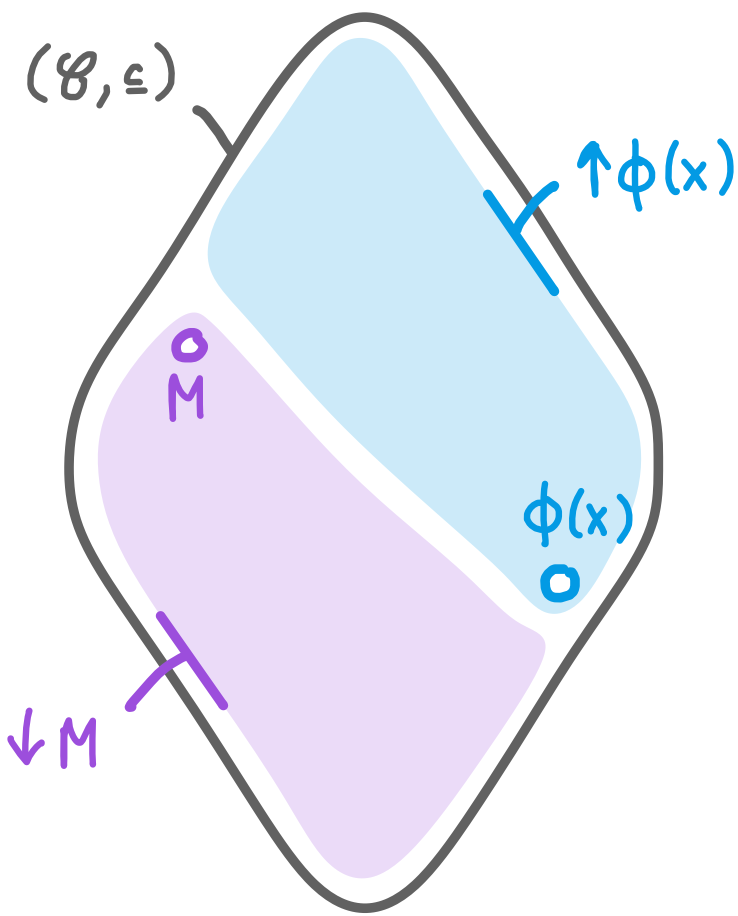
- Σ_C = e → ca, d → ba, b → a, c → a,
  abcd → e, abce → d, af → bcde

  Σ_E = e → ca, d → ba, b → a, c → a,
  cd → e, be → d, af → bc

  af → de is not true in Σ_E

PROB: the E-base does not generate enough elements

DEF: $x \in S$ is prime in $(S, \mathscr{C})$ if it has no minimal generators of size ≥ 2.



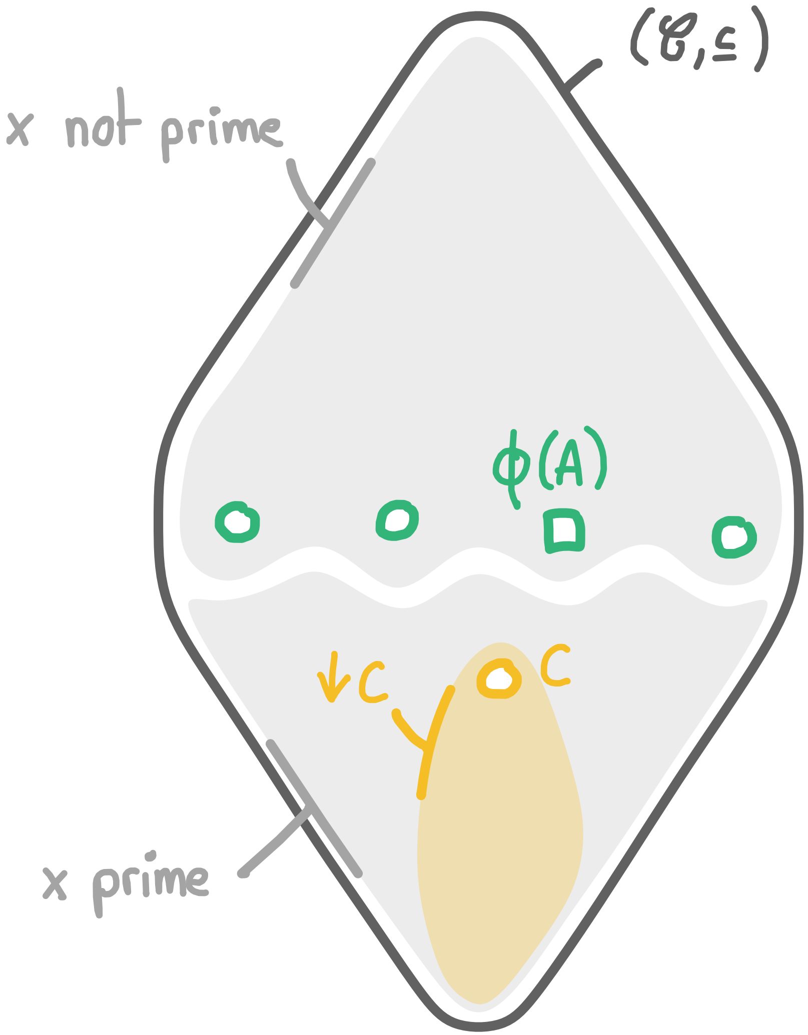
x is prime

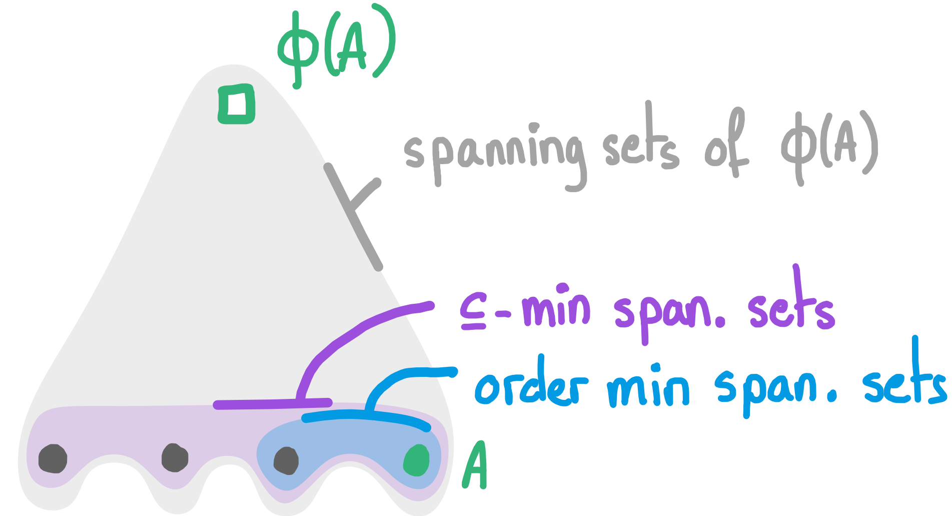
$\Leftrightarrow$ x has no E-generators

$\Leftrightarrow$ $\phi(x)$ is join-prime in $(\mathscr{C}, \subseteq)$

$\Leftrightarrow$ there is a unique maximal closed set M in $\{C : C \in \mathscr{C}, x \notin C\}$

$\mathcal{E}$-generators and primality





LEM: $A \subseteq S$ $\mathcal{E}$-gen of x iff:

(1) $x \in \phi(A)$, $x \notin \phi(a)$ for $a \in A$

(2) for $C \in \mathcal{C}$, $x \in C$ and $C \subset \phi(A)$

$\Rightarrow$ x prime in $(C, \downarrow C)$

(3) A is an order-min span set of $\phi(A)$

The E-base reflects in the canonical base

IDEA: closures of E-gen of x delineate the part of the lattice where x is not prime. They are "essential" to the closure system and in fact essential strictly speaking.

THM (Adaricheva, V., 25+): for any $C \in \mathcal{C}$ that is the closure of some E-gen of x, there is $P \to C \setminus P$ in Σ_C and a E-gen A of x s.t. $A \subseteq P$ and $\phi(A) = \phi(P) = C$.

Question: what makes things work in semidistributive lattices?

they are built around prime elements!

- semidistributivity = join- + meet- semidistributivity ($SD_j + SD_m$)

- SD_j says: each $C \in \mathcal{C}$ has a unique order min. span. set that moreover consists in prime elements of $\downarrow C$

- SD_m adds: each pseudo-closed set P reduces to a joint E-gen of enough prime elements of predessors of $\phi(P)$

E-base ⊆ D-base ⊆ Canonical direct base

order + closure min. — order min. — subset min.

⚠ : unlike the D-base and the canonical direct base, the E-base is not always valid

Playing with prime elements, quasi-closed, pseudo-closed and essential sets we can show that:

- the E-base reflects in the canonical base
- the E-base of SD lattices is valid and minimum

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