Hierarchical decompositions of dihypergraphs

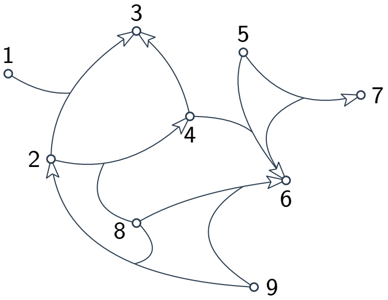
ICTCS 2020

Lhouari Nourine and *Simon Vilmin*.

LIMOS, UCA

September 2020

Dihypergraphs

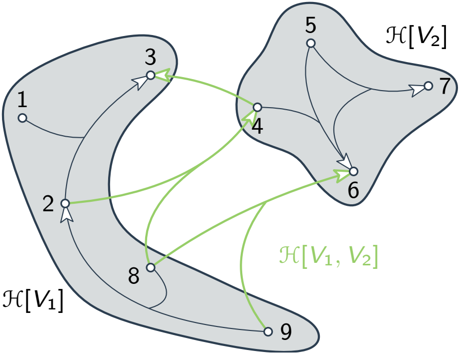


- Vertex set $\mathsf{V} = [9]$,
- *directed* hypergraph (*dihypergraph*) $\mathcal{H} = (\mathsf{V}, \mathcal{E})$ with *arcs* $\mathcal{E} = \{(12, 3), (4, 3), (45, 6), (56, 7), (28, 4), (89, 2), (89, 6)\}$.

Dihypergraph decomposition

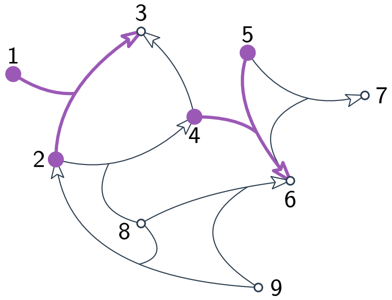
- Dihypergraphs :
  - an arc (B, h) is made of a *body* B and a *head* h,
  - sometimes known as B-graphs [Ausiello, Luigi, 2017], [Gallo et al., 1993],
  - applications in various fields of computer science [Ausiello et al., 1986], [Bertet et al., 2018].
- Decomposition strategy :
  - *split* : bipartition V_1, V_2 of V that cuts $\mathcal{H}$ in two disjoint parts $\mathcal{H}[V_1], \mathcal{H}[V_2]$ interacting together through a *bipartite dihypergraph* $\mathcal{H}[V_1, V_2]$,
  - recursive application of the splitting operation rises a *$\mathcal{H}$-tree*, a *hierarchical decomposition* of $\mathcal{H}$.

Split



How to find a split

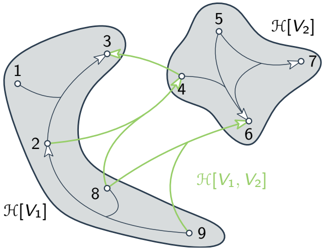
- 1 and 2 are *body-connected*
- 1289 is a *body-connected* component of $\mathcal{H}$

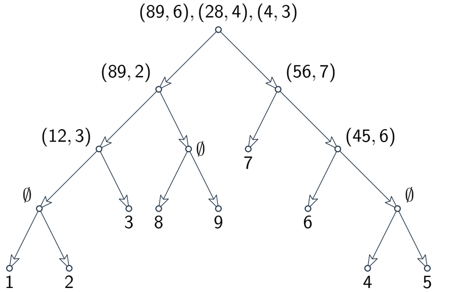


Theorem (Nourine, V., 2020+)

Let $\mathcal{H} = (V, \mathcal{E})$ be a dihypergraph. There is a split for $\mathcal{H}$ if and only if it is not body-connected.

Hierarchical Decomposition



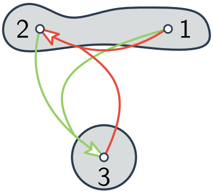


Hierarchical Decomposition

Theorem (Nourine, V., 2020+)

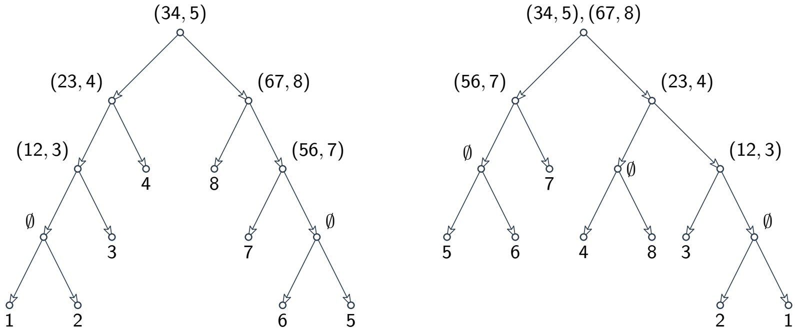
Let $\mathcal{H}$ be a dihypergraph. There is an algorithm which computes a $\mathcal{H}$-tree for $\mathcal{H}$, if it exists, in polynomial time and space in the size of $\mathcal{H}$.

- Dihypergraph without splits : $\mathcal{H} = ([3], \{(12, 3), (13, 2)\})$.



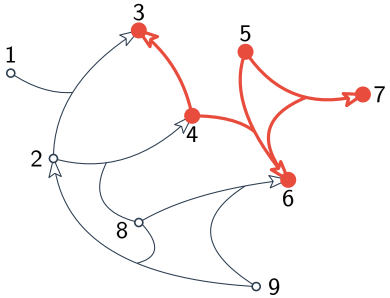
- Decomposition to *$\mathcal{H}$-factors* instead.

Multiple $\mathcal{H}$-trees



- $\mathcal{H} = (V, \mathcal{E})$, $V = [9]$ and $\mathcal{E} = \{(12, 3), (23, 4), (34, 5), (56, 7), (67, 8)\}$
-

Dihypergraph, closure systems



- Let $F = 45$, it *fails* $(45, 6)$ and $(4, 3)$3456 fails $(56, 7)$34567 fails no other arc
- We add 36 to FWe add 7So $F^{\mathcal{H}} = 34567$

Dihypergraph and closure systems

- $\mathcal{F}_{\mathcal{H}} = \{F^{\mathcal{H}} \mid F \subseteq \mathrm{V}\}$ is a *closure system* (ordered by $\subseteq$):
  - $\mathrm{V} \in \mathcal{F}_{\mathcal{H}}$,
  - $F_1, F_2 \in \mathcal{F}_{\mathcal{H}}$ implies $F_1 \cap F_2 \in \mathcal{F}_{\mathcal{H}}$.

- Let $X \subseteq \mathrm{V}$, the *trace* of $\mathcal{F}_{\mathcal{H}}$ is $\mathcal{F}_{\mathcal{H}} : X = \{F \cap X \mid F \in \mathcal{F}_{\mathcal{H}}\}$.

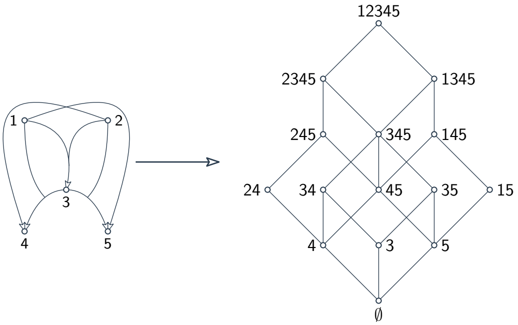
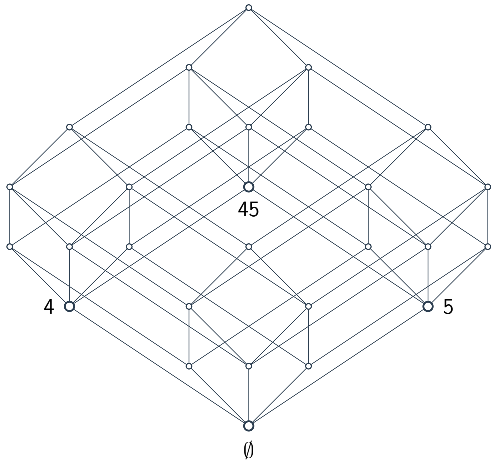


Figure: A dihypergraph $\mathcal{H} = (\mathrm{V}, \mathcal{E})$ with $\mathrm{V} = [5]$, $\mathcal{E} = \{(12, 3), (2, 4), (1, 5), (13, 4), (23, 5)\}$ and its corresponding $\mathcal{F}_{\mathcal{H}}$.

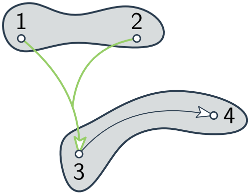
Case of empty split

- $\mathcal{H}[V_1, V_2]$ has *no* arcs
- $F_2 \in \mathcal{F}_2$ combined with any $F_1 \in \mathcal{F}_1$copies of $\mathcal{F}_1$ on each $F_2 \in \mathcal{F}_2\mathcal{F}$ direct product of $\mathcal{F}_1, \mathcal{F}_2$

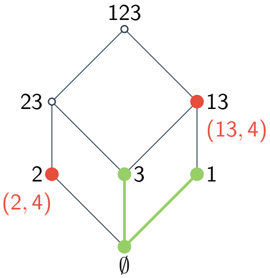


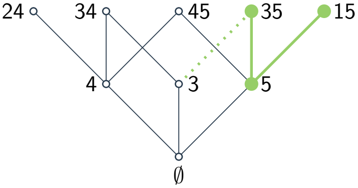
Acyclic split

- Restriction of the bipartite dihypergraph $\mathcal{H}[V_1, V_2]$
- *Acyclic* split: each arc (B, h) of $\mathcal{H}[V_1, V_2]$ goes from V_1 to V_2
- Expression of $\mathcal{F}$ in terms of $\mathcal{F}_1, \mathcal{F}_2$



Acyclic split on closure systems





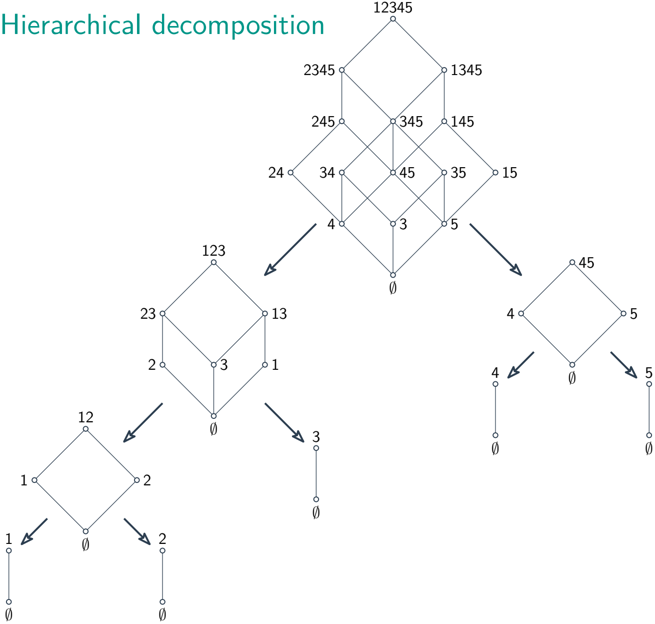
Trace decomposition of closure systems

Theorem (Nourine, V., 2020+)

Let $\mathcal{H} = (V, \mathcal{E})$ be a dihypergraph and (V_1, V_2) a split of $\mathcal{H}$. Then

1. $\mathcal{F}_{\mathcal{H}} \subseteq \mathcal{F}_1 \times \mathcal{F}_2$,
2. if $\mathcal{H}[V_1, V_2]$ has no edges, then $\mathcal{F}_{\mathcal{H}} = \mathcal{F}_1 \times \mathcal{F}_2$,
3. if the split is acyclic, then $\mathcal{F}_{\mathcal{H}} \colon V_i = \mathcal{F}_i, i \in \{1, 2\}$.

(Acyclic) Hierarchical decomposition



Splits for closure systems

- Characterization of some classes of lattice (Tamari lattices)
- Algorithm for translating between representations of closure systems. [Nourine, V., 2020]
- Algorithms for minimization ? New classes of closure systems ?

Conclusion

- Study of a partition operation, the *split* which divides a dihypergraph into two subhypergraph interacting via a bipartite one.
- Induces a lossless hierarchical decomposition, found in *polynomial time*.
- Reflects a decomposition of the underlying closure system.
- Further work on both the structure of $\mathcal{H}$-tree and closure systems.

Thank you for your attention!

References

- G. Ausiello, A. D'Atri, and D. Sacca.
  Minimal representation of directed hypergraphs.
  SIAM Journal on Computing, 15 :418-431, 1986.
- G. Gallo, G. Longo, and S. Pallottino, and S. Nguyen.
  Directed hypergraphs and applications,
  Discrete Applied Mathematics, 42 :177-201, 1993.
- G. Ausiello, L. Laura.
  Directed hypergraphs: Introduction and fundamental algorithms—a survey.
  Theoretical Computer Science, 658 :293-306, 2017.
- L. Nourine, S. Vilmin.
  Dihypergraph decomposition: application to closure system representations
  FCA4AI 2020, 31, 2020.
- K. Bertet, C. Demko, J.-F. Viaud, and C. Guérin.
  Lattices, closures systems and implication bases: A survey of structural aspects and algorithms.
  Theoretical Computer Science 743 :93-109, 2018.